VIBRATIONAL EQUATIONS OF VARIABLE MASS LAYERS AND MEMBRANES

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Vibrations of layers and membranes consisting of variable mass particles [1, 2], are considered. Adjacent and separated particles are classified. Differential equations and natural boundary conditions of vibrations of variable mass layers and membranes are derived. Two examples for vibrations are integrated by means of separation of variables.

1. Adjacent and separated particles can be of four types. To the first class belong particles adjacent and separated at each point of an elastic body surface and forming with them a single, continuous medium. To the second class belong adjacent and isolated particles at each point of a body surface, which are connected to the body, interacting with it and not among themselves. The third class consists of particles, adjacent and separated at each point of a finite number of regular curves of finite length, belonging to an elastic body surface. These particles interact with the elastic body, but not with each other. To the fourth class belong particles adjacent and separated in a finite number of points of a body.

Layer and membrane vibrations are considered for a time interval, during which they remain variable-mass elastic bodies of the same category as in the initial moment (mass adjacency and separation do not affect those features of the elastic body under consideration, by which both layers and membranes are classified).

It is assumed that a variable mass neutral layer does not affect its position relative to other layers. Particles can be adjacent and separated on both sides of the layers and membranes. Adjacent and separated particles move at the same time perpendicularly to the neutral layer and the membranes surface.

In the nondeformed state the neutral layer and membranes is a bounded region G with piecewise smooth boundaries L. The plane in which the region G is located is taken as the xy plane. The points in which fourth-type particle adjacency and separation occur are inside the region G, and are denoted by M_i or M_i (x_i, y_i). If such a point lies on the boundary L and has an arc coordinate s_j, it is denoted by M_j or M_j (s_j) $(i=1, 2, \ldots, k;\ j=k+1, \ldots, n)$.

The curve at whose points there occurs third-type particle adjacency and separation, not having common parts with other such curves and boundaries, except, perhaps, a finite or countable number of points, is denoted by $L_m(x_m, y_m)$, where x_m, y_m are the coordinates of this curve $(m=1, 2, \ldots, l)$. The parts of the boundary L, at whose points third-type particle adjacency and separation occur, are denoted by the symbol L_0.

We determine the intensity of reacting forces of adjacent and separated particles in their absolute R_a and relative R_r motions. For first type particles

$$R_{1a}=\frac{\partial\rho_{1r}{}^{+}}{\partial t}v_{1r}{}^{+}+\frac{\partial\rho_{1r}{}^{-}}{\partial t}v_{1r}{}^{-}+\frac{\partial\rho_{1l}{}^{+}}{\partial t}v_{1l}{}^{+}+\frac{\partial\rho_{1l}{}^{-}}{\partial t}v_{1l}{}^{-} \tag{1.1}$$

$$R_{1r}=\frac{\partial\rho_{1r}{}^{+}}{\partial t}\left(v_{1r}{}^{+}-\frac{\partial z}{\partial t}\right)+\frac{\partial\rho_{1r}{}^{-}}{\partial t}\left(v_{1r}{}^{-}-\frac{\partial z}{\partial t}\right)+\frac{\partial\rho_{1l}{}^{+}}{\partial t}\left(v_{1l}{}^{+}-\frac{\partial z}{\partial t}\right)+\frac{\partial\rho_{1l}{}^{-}}{\partial t}\left(v_{1l}{}^{-}-\frac{\partial z}{\partial t}\right)$$

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Here $\rho(x, y, t) = \rho_1^\circ + \rho_{1r}^+ + \rho_{1r}^- + \rho_{1l}^+ + \rho_{1l}^- \geq 0$ is the density at point M (x, y) at moment t; ρ_1° (x, y) $\geq$ 0 is the initial density at the point M(x, g); $\rho_{1r}^\pm$, $\rho_{1l}^\pm$, $v_{1r}^\pm$, $v_{1l}^\pm$ are the density and absolute velocities of adjacent and separated particles at point M(x, y) at moment t; and z(x, y, t) is the transverse displacement of the point M(x, y) of the neutral layer at moment t. The plus and minus indices indicate adjacency and separation, and the indices r and l denote particle motion in the positive and negative directions, respectively, of the z axis. At the same time

$$\rho_{1r}\pm(x, y, 0) = \rho_{1l}\pm(x, y, 0) = 0, \ \frac{\partial \rho_{1r}^+}{\partial t} \geqslant 0, \quad \frac{\partial \rho_{1l}^+}{\partial t} \geqslant 0,$$
$$\frac{\partial \rho_{1r}^-}{\partial t} \leqslant 0, \quad \frac{\partial \rho_{1l}^-}{\partial t} \leqslant 0$$

The intensity of reacting forces R_{1a} and R_{1r} in the absolute and relative motions are related by

$$R_{1a} = R_{1r} + \frac{\partial \rho_1}{\partial t}\frac{\partial z}{\partial t}$$

In this notation of reacting forces the lower indices a and r denote absolute and relative motion, respectively.

The same notation as in (1.1) is retained for particle adjacency and separation of the remaining types. The lower indices for the corresponding quantities of second-type particles is 2, for third-type particles L_0 and L_m, and for fourth-type particles M_i and M_j. Partial derivatives of fourth-type particle densities should be replaced by ordinary ones.

For first-and second-type particles the mass density and reactive forces intensity are the mass and force per unit area. For third-type particles these quantities refer to unit length, and for fourth-type particles these mass and force are concentrated at the corresponding point. In what follows it is convenient to consider the mass density and reactive force intensity of third- and fourth-type particles for internal points of the region G as the masses and forces relating to unit area, and for points of the boundary L as masses and forces relating to unit length.

Consequently, the surface density ρ_G of internal points of the region G and the linear density ρ_L of points of the boundary L equal at time t

$$\begin{aligned}\rho_G(x, y, t) &= \rho_1(x, y, t) + \rho_2(x, y, t) + \sum_{i=1}^{k} \rho_{M_i}(x_i, y_i, t)\,\sigma_1(x - x_i) \times \\ &\times \sigma_1(y - y_i) + \sum_{m=1}^{l} \rho_{L_m}(x_m, y_m, t)\,\sigma_1(x - x_m)\,\sigma_1(y - y_m) \\ \rho_L(s, t) &= \rho_{L_0} + \sum_{j=k+1}^{n} \rho_{M_j}(s_j, t)\,\sigma_1(s - s_j)\end{aligned} \tag{1.2}$$

where σ_1 is the first-order impulse function.

Similarly one determines the reactive force intensity R_{Ga} and R_{Gr} of absolute and relative motion of internal points of region G and the reactive force intensity R_{La} and R_{Lr} of points of the boundary L. For this one must replace the symbol ρ in (1.2) by R and add the lower index a or r. The following dependence holds among R_{Ga} and R_{Gr}, R_{La} and R_{Lr}:

$$R_{Ga} = R_{Gr} + \frac{\partial \rho_G}{\partial t}\frac{\partial z}{\partial t}, \qquad R_{La} = R_{Lr} + \frac{\partial \rho_L}{\partial t}\frac{\partial z}{\partial t}$$

The vibrational differential equations and boundary conditions are derived from the stationary Hamilton–Ostrogradskii action principle

$$\int_{t_0}^{t_1} [\delta(T - U) + \delta A]\,dt = 0 \tag{1.3}$$

Here T is the elastic body kinetic energy, U is the potential energy of elastic forces, and δA is the elementary work of reactive forces on possible displacements

$$\delta A = \iint_G R_{Ga}\,\delta z\,dx\,dy + \oint_L R_{La}\,\delta z\,ds$$

where t_0 and t_1 are fixed moments at which z is not varied. The region G is also not varied.

2. The kinetic and potential energies of variable mass layer are determined by the relations

$$T = \frac{1}{2}\iint_G \rho_G \left(\frac{\partial z}{\partial t}\right)^2 dx\,dy + \frac{1}{2}\oint_L \rho_L \left(\frac{\partial z}{\partial t}\right)^2 ds$$

$$U = \frac{1}{2}\iint_G D\left\{(\Delta z)^2 - 2(1-\sigma)\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy$$

Substituting these expressions in (1.3) and performing some transformations [3], we obtain the equation of free transverse vibrations of a variable mass layer. The equation can be represented in two forms, the first of which is conveniently used for known reactive forces of absolute motion of adjacent and separated particles, and the second for known reactive forces of relative motion

$$\frac{\partial}{\partial t}\left(\rho_G \frac{\partial z}{\partial t}\right) + \Delta(D\Delta z) - (1-\sigma)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 z}{\partial x^2}\right) = R_{Ga} \tag{2.1}$$

$$\rho_G \frac{\partial^2 z}{\partial t^2} + \Delta(D\Delta z) - (1-\sigma)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 z}{\partial x^2}\right) = R_{Gr} \tag{2.2}$$

Here z(x, y, t), d(x, y, t), E = const, σ = const are the displacement cylinder tension, elastic modulus, and Poisson coefficient, respectively, and Δ is the Laplacian.

Also obtained from (1.3) is the boundary condition

$$\oint_L \left(N_L - \frac{\partial H_L}{\partial s}\right)\delta z\,ds - \oint_L M_L \frac{\partial(\delta z)}{\partial \nu} ds = 0 \tag{2.3}$$

where M_L, H_L, N_L are the bending moment, torque, and transverse force on the layer contour, determined by the relations

$$M_L = -D\left[\left(\frac{\partial^2 z}{\partial x^2} + \sigma\frac{\partial^2 z}{\partial y^2}\right)\cos^2\theta + \left(\frac{\partial^2 z}{\partial y^2} + \sigma\frac{\partial^2 z}{\partial x^2}\right)\sin^2\theta + (1-\sigma)\frac{\partial^2 z}{\partial x\,\partial y}\sin 2\theta\right] \tag{2.4}$$

$$H_L = \frac{(1-\sigma)D}{2}\left[\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right)\sin 2\theta + 2\frac{\partial^2 z}{\partial x\,\partial y}\cos 2\theta\right] \tag{2.5}$$

$$N_L = N - \frac{\partial(D\Delta z)}{\partial \nu} + (1-\sigma)\left[\left(\frac{\partial D}{\partial y}\frac{\partial^2 z}{\partial x^2} - \frac{\partial D}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y}\right)\sin\theta + \left(\frac{\partial D}{\partial x}\frac{\partial^2 z}{\partial y^2} - \frac{\partial D}{\partial y}\frac{\partial^2 z}{\partial x\,\partial y}\right)\cos\theta\right] \tag{2.6}$$

where

$$N = \frac{\partial}{\partial t}\left(\rho_L \frac{\partial z}{\partial t}\right) - R_{La} = \rho_L \frac{\partial^2 z}{\partial t^2} - R_{Lr}$$

Here ν is a unit vector external normal to the boundary L, and θ is the angle between the unit vector ν and the positive direction of the abscissa axis.

For layers whose widths are identical at all points (they depend only on time) the derivative of the cylindric tension D with respect to coordinates do not appear in the vibrational equations and boundary conditions.

Consider an example. First type particles with vanishing relative velocities are adjacent to a uniform rectangular layer, freely supported by a contour L (x = 0, x = a, y = 0, y = b). The layer width is identical at all points and varies by the law $h = e^t$, and the bulk density of the material ρ is constant. In the initial moment

$$z|_{t=0} = f_1(x, y), \qquad \left.\frac{\partial z}{\partial t}\right|_{t=0} = f_2(x, y)$$

It follows from the conditions of the problem that

$$\rho_G = \rho e^t, \quad D = \frac{E}{12(1-\sigma^2)} e^{3t}, \quad R_{Gr} = 0$$

For the case considered, the vibrational equation (2.2) acquires the form

$$\frac{\partial^2 z}{\partial t^2} + \alpha^2 e^{2t} \Delta(\Delta z) = 0, \qquad \alpha^2 = \frac{E}{12\rho(1-\sigma^2)}$$

As a result of separating variables, determining the characteristic forms [4], and solving the frequency equation, we obtain

$$z(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} [A_{ij} I_0(\lambda_{ij} e^t) + B_{ij} Y_0(\lambda_{ij} e^t)] \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}$$
$$\left(i, j = 1, 2, 3, \ldots; \ \lambda_{ij} = \alpha\pi^2 \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]\right)$$

I_0, I_1, Y_0, Y_1 are first and second kind Bessel functions. Using the well-known relation

$$I_1(u) Y_0(u) - I_0(u) Y_1(u) = 2/u$$

we find

$$A_{ij} = -\frac{2\lambda_{ij}}{ab} \int_0^a \int_0^b \left[f_1 Y_1(\lambda_{ij}) + f_2 \frac{Y_0(\lambda_{ij})}{\lambda_{ij}} \right] \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \, dx\, dy$$

$$B_{ij} = \frac{2\lambda_{ij}}{ab} \int_0^a \int_0^b \left[f_1 I_1(\lambda_{ij}) + f_2 \frac{I_0(\lambda_{ij})}{\lambda_{ij}} \right] \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \, dx\, dy$$

For sufficiently large u the functions $I_0(u)$, $I_1(u)$, $Y_0(u)$, $Y_1(u)$ are of order $u^{-1/2}$, leading to the inequality

$$| A_{ij} I_0(\lambda_{ij} e^t) + B_{ij} Y_0(\lambda_{ij} e^t)| < M e^{-t/2} / ij \quad (M = \text{const})$$

This estimate indicates that the series obtained converges uniformly.

In the given case adjacent particles do not appear in the characteristic forms of layer vibrations, but leads to damping of its free vibrations despite the absence of inelastic resistance forces.

3. The kinetic and potential membrane energies are determined by the relations

$$T = \frac{1}{2} \iint_G \rho_G \left(\frac{\partial z}{\partial t}\right)^2 dx\, dy + \frac{1}{2} \oint_L \rho_L \left(\frac{\partial z}{\partial t}\right)^2 ds$$
$$U = \frac{1}{2} \iint_G T_0 \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dx\, dy$$

where T_0 is the membrane tension.

The vibrational equations and boundary conditions are found by substituting the kinetic and potential energy expressions into (1.3). They are written in two forms, one corresponding to absolute motion of adjacent and separated particles, and the second to relative motion. The membrane-vibrational equations are

$$\frac{\partial}{\partial t}\left(\rho_G \frac{\partial z}{\partial t}\right) - \frac{\partial}{\partial x}\left(T_0 \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(T_0 \frac{\partial z}{\partial y}\right) = R_{Ga} \tag{3.1}$$

$$\rho_G \frac{\partial^2 z}{\partial t^2} - \frac{\partial}{\partial x}\left(T_0 \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(T_0 \frac{\partial z}{\partial y}\right) = R_{Gr} \tag{3.2}$$

with boundary conditions

$$\frac{\partial}{\partial t}\left(\rho_L \frac{\partial z}{\partial t}\right) + T_0 \frac{\partial z}{\partial \nu} = R_{La} \tag{3.3}$$

$$\rho_L \frac{\partial^2 z}{\partial t^2} + T_0 \frac{\partial z}{\partial \nu} = R_{Lr} \tag{3.4}$$

Consider an example. First type separated particles leave a membrane closed by a contour L (x=0, x= a, y=0, y=b), so that

$$T_0 = \text{const}, \ \rho_{1r}^+ = \rho_{1e}^+ = 0, \ \rho_{1r}^- = \rho_{1l}^- = {}^1/_2\, \rho_1{}^0 (e^{-2t} - 1),$$
$$v_{1r}^- + v_{1e}^- = \partial z / \partial t$$

The initial conditions are

$$z\,|_{t=0} = f_1(x, y), \quad \frac{\partial z}{\partial t}\Big|_{t=0} = f_2(x, y)$$

The membrane vibrational equations are reduced to the form

$$\frac{\partial^2 z}{\partial t^2} - \frac{\partial z}{\partial t} - \alpha^2 e^{2t} \Delta z = 0, \qquad \alpha^2 = \frac{T_0}{\rho_1{}^\circ}$$

The solution is found by separation of variables

$$z(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} [A_{ij} \cos(\lambda_{ij} e^t) + B_{ij} \sin(\lambda_{ij} e^t)] \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$
$$\left(i, j = 1, 2, 3, \ldots, \lambda_{ij} = \alpha\pi \sqrt{\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2}\right)$$
$$A_{ij} = \frac{4}{ab} \int_0^a \int_0^b \left(f_1 \cos \lambda_{ij} - \frac{f_2}{\lambda_{ij}} \sin \lambda_{ij}\right) \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$
$$B_{ij} = \frac{4}{ab} \int_0^a \int_0^b \left(f_1 \sin \lambda_{ij} + \frac{f_2}{\lambda_{ij}} \cos \lambda_{ij}\right) \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$

Uniform convergence of the series obtained follows from

$$|A_{ij} \cos(\lambda_{ij} e^t) + B_{ij} \sin(\lambda_{ij} e^t)| < M / ij \qquad (M = \text{const})$$

It follows from the solution that the characteristic vibrational forms of the system considered coincide with the vibrationalforms of a constant mass membrane. The eigenfrequencies differ by a factor e^t. Particle separation is manifested in the given case by membrane eigenfrequencies continuously increasing with time.

We note the characteristics of the vibrational equations of variable mass layers and membranes which differ from the equations of classical theory of layer and membrane vibrations. The coefficients of these equations can be functions not only of coordinates, but also of time. In this case one must add to external loads reactive forces of adjacent and separated particles. These features of the vibrational equations of variable mass layers and membranes can lead to solutions differing from those of the vibrational equations of constant mass layers and membranes.

LITERATURE CITED

1. N. A. Slezkia, "On the deformation mechanism of media with variable mass particles," Vestn. MGU, No. 10 (1951).
2. N. A. Slezkia, "Basic equations of motion of a deformable medium with variable mass particles," Dokl. Akad. Nauk, 79, No. 1 (1951).
3. I. M. Gel'fand and S. V. Fomin, Calculus of Variations, Prentice-Hall (1963).
4. A. P. Filippov, Vibrations of Deformable Systems [in Russian], "Mashinostroenie," Moscow (1970).